Fundamental burn-up mode in a pebble-bed type reactor

Xue-Nong Chen*, Edgar Kieffhaber, Werner Maschek

Institute for Nuclear and Energy Technologies, Forschungszentrum Karlsruhe, Hermann-von-Helmholtz-Platz 1, D-76344 Eggenstein-Leopoldshafen, Germany

Abstract

This paper deals with a pebble-bed type reactor, in which the fuel is loaded from one side (top) and discharged from the other side (bottom). A boundary value problem of a single group diffusion equation coupled with simplified burn-up equations is studied, where the natural radioactive decay processes are neglected in the burn-up modelling. An asymptotic burning wave solution is found analytically in the one-dimensional case, which is called as fundamental burn-up mode. Among this solution family there are two particular cases, namely, a classic fundamental solution with a zero burn-up and a partial solitary burn-up wave solution with a highest burn-up. An example of Th–U conversion is considered and the solutions are presented in order to show the mechanism of the burning wave.

Keywords: CANDLE burn-up concept; Pebble-bed reactor; Diffusion model; Burn-up equations; Fission wave; Solitary wave

1. Introduction

It has been reported by Feoktistov (1989) that there are self-propagating nuclear burning waves in fertile media of $^{238}$U and $^{232}$Th. This mechanism was proposed by Teller et al. (1996) in some detail for an interesting concept of a self-controlled nuclear fission reactor, in which two nuclear breeding/burning waves are ignited by an initially critical neutron flux hump in the core center and then slowly propagate away from the center in two opposite axial directions. Natural thorium and uranium fuels can be used and a high fuel burn-up can be achieved by this type of reactor. Consequently no fuel enrichment and reprocessing are needed, and a long operation duration is possible without refueling interruption.

Fundamental understanding of this new type of reactor was provided by Van Dam (2000a,b) and Seifritz (2000), where solitary wave solutions are obtained from a single group diffusion equation with burn-up dependent macroscopic coefficients, which are either suitably assumed or solved from the burn-up equations of a realistic conversion chain. Moreover, multi-dimensional and feedback effects can be taken into account as well (Chen and Maschek, 2005; Chen et al., 2005, 2006). Intensive numerical studies of multi-group diffusion and burn-up coupled equations were carried out for this kind of reactor by Sekimoto and Ryu (2000), Sekimoto et al. (2001), Ohoka and Sekimoto (2004) and Sekimoto and Miyashita (2005). The feasibility of this new concept by achieving a quasi-asymptotic solution was numerically demonstrated. This concept was coined “CANDLE” burn-up strategy.

In contrast to the above CANDLE concept, in the pebble-bed type reactor the fuel is moving instead of moving neutron flux. Evidently, due to the relative motion principle, these two kinds of motions are equivalent. Therefore, the pebble-bed type reactor is an interesting subject to be investigated, for which the CANDLE concept can be applied. As already known, the chosen fuel has to have two conditions for the existence of a solitary wave: the fresh fuel should be subcritical, i.e. $k_\infty < 1$ at $\psi = 0$, and $k_\infty$ should first increase with neutron fluence $\psi$ into the supercritical range and then decrease again to the subcritical range. Technically this can be realized by adding burnable poisons to the fuel of a high temperature gas cooled reactor (HTGR). This was shown by numerical simulations in a one-dimensional case by Van Dam (2000a) and in a two-dimensional case by Ohoka and Sekimoto (2004). Nevertheless the transition from a conventional pebble-bed reactor to the
The CANDLE concept (the solitary burn-up wave solution) has not been demonstrated theoretically, because, in the conventional pebble-bed reactor, the active core length is finite, the fresh fuel is supercritical and $k_{\infty}$ is monotonically decreasing with the neutron fluence.

In this paper we deal with the boundary value problem of the diffusion equation coupled with simplified burn-up equations and find that this problem is analytically solvable in the one-dimensional case. Thus, we obtain a more general solution, which we call fundamental burn-up mode. The solitary burn-up wave is only a particular case in this kind of solution. For a given composition of fresh fuel the fuel moving speed and the core length are related in order to satisfy the criticality condition. The lowest speed corresponds to a solitary wave solution where the core length is theoretically infinite and the burn-up achieves its maximum. As an example the $^{232}$Th–$^{233}$U conversion chain is chosen and results are presented. The analytic solution obtained in this paper can be used for the purpose of burn-up optimisation.

2. Neutronic model

2.1. Diffusion equation

For the sake of simplicity we consider a one-dimensional single group steady state problem without any external source. For the neutron balance in the core the diffusion equation reads

$$\frac{d}{dx} \left( D \frac{d \phi}{dx} \right) + \nu \Sigma_f \phi - \Sigma_a \phi = 0,$$  \hspace{1cm} (1)

where $\phi$ is the neutron flux, $D$ the diffusion coefficient, $\nu$ the average number of generated neutrons per fission, $\Sigma_a$ and $\Sigma_f$ are the macroscopic absorption and fission cross-sections, respectively. In the above equation a convective term due to the fuel motion has been neglected, because the ratio of the fuel moving speed to the neutron average velocity is negligibly small.

Unlike the CANDLE reactor that might be theoretically infinitely long, the pebble-bed reactor has a finite length. Therefore we have to pose a suitable boundary condition for the pebble-bed reactor problem. For a bare core, i.e. a core without reflectors, it reads

$$\phi + d \frac{d \phi}{dn} = 0, \quad \text{at } x = 0 \text{ and } l, \quad \text{with } d = \frac{2}{3 \Sigma_i},$$  \hspace{1cm} (2)

where $n$ is the outward normal vector on the boundary, $d$ the extrapolation distance and $\Sigma_i$ is the macroscopic transport cross-section. Suppose the fuel is moving from left to right with a speed $u$. This means the fresh fuel is fed in at $x = 0$ and the burned fuel unloaded at $x = l$, see Fig. 1.

The coefficients in Eq. (1) are burn-up dependent rather than constant. Mathematically it is a typical Sturm–Liouville eigenvalue problem. The zero eigenvalue case corresponds to the so-called critical state, which can be achieved by appropriately choosing fuel composition and core size.

2.2. Burn-up equations

We consider a truncated $^{232}$Th–$^{233}$U conversion chain for our burn-up calculation assuming a thermal neutron spectrum. This means that only the heavy metals $^{232}$Th, $^{233}$U, $^{234}$U and $^{235}$U, characterized by the indices $i = 2, 3, 4$ and 5, and, in addition, a typical burnable fission product pair (FPP) are taken into account. Because the radioactive decay processes are, in the considered case, either too short or too long with respect to the considered time scale of the order of several years, natural radioactive decay processes and $(n, 2n)$ processes are neglected. Thus, the simplified burn-up equations can be written for the conversion chain shown in Fig. 2 as

$$\frac{\partial N_i}{\partial t} = -N_i \sigma_{a,i} \phi + N_{i-1} \sigma_{c,i-1} \phi, \quad i = 3, 4, 5,$$  \hspace{1cm} (3)

$$\frac{\partial N_{\text{FPP}}}{\partial t} = -N_{\text{FPP}} \sigma_{\text{FPP}} \phi + \sum_{i=2,3,4,5} N_i \sigma_{f,i} \phi,$$  \hspace{1cm} (3)

where $N_i$ is the atom number density of isotope $i$, $\sigma_{a,i}$, $\sigma_{c,i}$, $\sigma_{f,i}$ are the microscopic absorption, capture and fission cross-sections of isotope $i$, respectively.

2.3. Coupling of diffusion and burn-up equations

In the diffusion equation the macroscopic coefficients are neither uniform in space, nor constant in time. They depend in general on the material composition that changes with fuel burn-up. This means that the diffusion equation is coupled by the burn-up equations through the macroscopic coefficients $\Sigma_a$, $\Sigma_f$, $\Sigma_i$ and $D$ in the following manner,

$$\Sigma_a = \sum_i N_i \sigma_{a,i}, \quad \nu \Sigma_i = \sum_i N_i \nu \sigma_{f,i}, \quad \Sigma_f = \sum_i N_i \sigma_{f,i},$$  \hspace{1cm} (4)

$$D = \frac{1}{3 \Sigma_u}.$$  \hspace{1cm} (4)

The burn-up equations provide the macroscopic coefficients to the diffusion equation and the diffusion equation provides the neutron flux to the burn-up equations. Table 1 gives typical microscopic cross-sections averaged over a Maxwellian neutron spectrum. The given values should not be considered as

![Fig. 1. Schematic of a one-dimensional pebble-bed reactor.](image1)

![Fig. 2. Simplified Th–U conversion chain.](image2)
representative but may illustrate order of magnitude and tendencies, which is sufficient for this theoretical study.

3. Mathematical solution

3.1. Solution of burn-up equations

The burn-up Eq. (3) can be solved in a straightforward manner, either numerically or analytically. As in the case here, in which the natural radioactive decay processes are neglected, all atom number densities \( N_i \) can be expressed as functions of the neutron fluence \( \psi \), i.e.

\[
N_i = N_i(\psi), \quad \psi = \int_0^t \phi \, dt.
\] (5)

The fresh fuel is proposed to consist only of \(^{232}\text{Th}\) and \(^{233}\text{U}\) as heavy metal components. Therefore, the total initial heavy metal atom number density is \( N_0 = N_{2,0} + N_{3,0} \). A solution of the burn-up Eq. (4) is shown in Fig. 3 for an enrichment of 3\% (\( N_{3,0}/N_0 = 0.03 \)).

3.2. Solution of the diffusion equation

Because the fuel is moving and the neutron flux is stationary with respect to the core-fixed coordinate system, the fuel residence time \( t \) and the fuel position \( x \) have a relation as \( dx = u \, dt \), \( u \) is the fuel drift speed. Therefore, from Eq. (5) we have \( \psi(x) = \int_0^t \phi \, dx \) and the macroscopic net production cross-section \( f(\psi) = \nu \Sigma_t(\psi) - \Sigma_r(\psi) \) is a known function obtained from the solution of the burn-up equations.

Table 1

<table>
<thead>
<tr>
<th></th>
<th>(^{232}\text{Th})</th>
<th>(^{233}\text{U})</th>
<th>(^{234}\text{U})</th>
<th>(^{235}\text{U})</th>
<th>FPP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )</td>
<td>2.21</td>
<td>2.49</td>
<td>2.37</td>
<td>2.42</td>
<td>0</td>
</tr>
<tr>
<td>( \sigma_i [\text{barn}] )</td>
<td>0</td>
<td>468</td>
<td>0.407</td>
<td>505</td>
<td>0</td>
</tr>
<tr>
<td>( \sigma_e [\text{barn}] )</td>
<td>6.55</td>
<td>41.8</td>
<td>90.5</td>
<td>86.4</td>
<td>35.4</td>
</tr>
</tbody>
</table>

Now Eq. (1) can be rewritten as

\[
\frac{d}{dx} \left( D(\psi) \frac{d\phi}{dx} \right) + f(\psi)\phi = 0,
\] (6)

which is twice analytically integrable. The first integral is just carried out by integrating directly over \((0, x)\), where there is a contribution from the nontrivial boundary condition at \( x = 0 \). The second one is done by multiplying \( \phi \) to the first integral result and integrating it once more over \((0, x)\). Thus, by defining

\[
g(\psi) = \int_0^\psi f(\psi) \, d\psi, \quad h(\psi) = \int_0^\psi g(\psi) \frac{D(\psi)}{D(\psi)} \, d\psi,
\] (7)

the first integral gives

\[
\frac{d}{dx}\phi = \frac{D_0}{D(\psi)} \left( \frac{d}{dx}\phi \right)_{x=0} - \frac{g(\psi)}{D(\psi)},
\] (8)

and the second one gives

\[
\phi^2 = \phi_0^2 + 2u \left( \frac{d}{dx}\phi \right)_{x=0} E(\psi) - 2u^2 h(\psi).
\] (9)

From the viewpoint of dynamic systems, the solution has been already completed, since the solution can be displayed in the phase plane \((\phi, \psi)\) as parametric functions of \( \psi \) in Eqs. (8) and (9). In particular, \( \psi(x) \) can be determined from \( \psi(x) = \phi(\psi), \phi(x) \) can be obtained as well. The nontrivial solution obtained here is essential for the burn-up problem and can be called fundamental burn-up mode.

As usual \( k_{\infty} \) has to be larger than unity in some \( \psi \) range or some spatial region. This is a necessary condition to make the core be critical. If the core length \( l \) is given, the fuel drift speed \( u \) will be determined by the critical solution. We will demonstrate the solution in detail by an example in Section 3.4.

![Fig. 3. Solution of the burn-up equations with \( N_{2,0}/N_0 = 0.97 \) and \( N_{3,0}/N_0 = 0.03 \).](image)
3.3. Variables, parameters and their normalization

The spatial variables to be solved are the neutron flux $\psi$, the neutron fluence $\psi$ and the atom number density $N_i$ of nuclide $i$ in this problem. $D_0$, $\sigma_{a,0} = \Sigma_{a,0}/N_0$ and $N_0 = \Sigma_{i,0}$ are chosen as basic dimensional variables used for normalization, where subscript 0 refers to either core inlet or fresh fuel with $\psi = 0$. The freely chosen parameters might be the neutron flux $\phi_0$ at the core inlet and the fuel drift speed $u$, whereas the core length $l$ is a derived parameter. For the sake of easy recognition the corresponding capital letters will be used for the non-dimensional variables in the following.

A suitable normalization makes the formulation more clear and reduces the number of input parameters. The most suitable normalization of $\psi$ for this kind of problem was suggested by Seifritz (2000), $\Psi = \psi \sigma_{a,0}$ so that the non-dimensional fluence $\Psi$ is usually of order of one. A natural way to normalize $\phi$ is $\Phi = \phi/\phi_0$.

From the normalization of $\phi$ and $\psi$ a typical time scale is derived as $t_0 = 1/(\phi_0 \sigma_{a,0})$. The diffusion length can be used as the length scale: $l_0 = \sqrt{D_0/\Sigma_{a,0}}$. Therefore we can have a typical drift speed $u_0 = l_0/t_0 = (\phi_0/N_0) \sqrt{\Sigma_{a,0}/(3 \Sigma_{n,0})}$. The coordinate $x$ and the drift speed $u$ are normalized by its typical values $l_0$ and $u_0$ as $X = x/l_0$ and $U = u/u_0$. The expression of the above fuel drift speed $u_0$ implies that the dimensional fuel drift speed is directly proportional to $\phi_0$ and inversely proportional to $N_0$. The chosen values of the basic variables and their derived dimensional scales are given in (Table 2).

As a result of this normalization, e.g. the diffusion equation becomes

$$\frac{d}{dX} \left( \frac{D(\Psi)}{D_0} \frac{d\Phi}{dX} \right) + F(\Psi) \Phi = 0,$$


where $F$ is the normalized net production cross-section, i.e. $F = f/(N_0 \sigma_{a,0})$.

3.4. Coupled solution

The coupled solution is obtained by substituting $F(\Psi)$ obtained from the burn-up equation into the solution of the diffusion equation. Assume the macroscopic transport cross-section to be constant and assign it in particular be $\Sigma_n = c \Sigma_{a,0}$, where $c = 0.5$ is taken for the current example. Then $\Sigma = D_0 = 1/(3 \Sigma_n)$ and $d/l_0 = 2/3$ and the non-dimensional boundary conditions become $(\Phi_c)_0 = K \Phi_0$ and $(\Phi_x)_L = -K \Phi_x$, where $K = l_0/d$. The solution in Eqs. (8) and (9) in the non-dimensional form can be written as

$$\frac{d}{dX} \Phi = K - U G(\Psi), \quad \text{with} \quad G(\Psi) = \int_0^\Psi F(\Psi') d\Psi, \quad (11)$$

$$\Phi^2 = 1 + 2UK \Psi - 2U^2 H(\Psi), \quad \text{with} \quad H(\Psi) = \int_0^\Psi G(\Psi') d\Psi. \quad (12)$$

After normalization, only the non-dimensional drift speed $U$ and the core length $L$ are left in the problem, which will be determined by the solution. The criticality condition, which is actually an eigenvalue problem of a nonlinear equation, i.e. the nontrivial solution of the nonlinear diffusion equation, will provide a relation between $U$ and $L$. This means for a certain $U$, there is a value of $L$ to make the core critical. It is easy to see that there are two limit cases. One is $U$ being infinitely large, so that the whole core is filled with fresh fuel homogeneously. The solution in this case tends to be the fundamental mode of a homogeneous core, i.e. sin or cos-function. In this case the core is shortest and the burn-up is zero. The other case is that $U$ takes its minimum value and $L$ tends to infinity. In this case a partial solitary wave shape of neutron flux is formed and the burn-up reaches a maximum. For the solitary wave solution the following conditions at $X = \infty$ that corresponds to $\Psi = \Psi_{\text{max}}$ are satisfied

![Fig. 4. Solutions in the phase plane ($\Phi$, $\Phi_x$) and in the physical plane ($X$, $\Phi$) in the case of $N_{2,0}/N_0 = 0.97$ and $N_{3,0}/N_0 = 0.03$.](image-url)
\[ \Phi = 0 \text{ and } \frac{\partial}{\partial x} \Phi = 0, \text{ at } \Psi = \Psi_{\text{max}}. \]

This leads to the condition for the fully solitary wave solution, namely, \( h(\Psi) \) has a double zero at fluence \( \Psi_{\text{max}} \) in the particular case of \( \phi_0 = 0 \) in Chen et al. (2005).

The solution can be immediately presented in the phase plane \((\Phi, \Phi_x)\) by regarding \( \Phi \) and \( \Phi_x \) as parametric functions of \( \Psi \) in Eqs. (11) and (12). For several typical values of \( U \), the solutions are shown in the left plot of Fig. 4. The corresponding solutions in the physical plane \((X, \Phi)\) are shown in the right plot of Fig. 4. The boundary conditions at the two ends of core are represented as two straight lines in the phase plane and the solution starts from one line and ends at the other. The neutron fluxes at inlet and outlet boundaries are determined by cross points of the solution curve and the boundary lines. The relations between the fuel drift speed \( U \) and the core length \( L \) and between the burn-up and \( L \) can be obtained from the solution and are presented in Fig. 5. If the fuel drift speed is decreased, the core length \( L \) has to be increased, because otherwise the reactor would become subcritical. If the core length reaches infinity, the fuel drift speed will have a minimum, a solitary neutron flux shape will be formed and the burn-up will have a maximum. In the present case the minimum of the drift speed is found to be \( U_{\text{min}} = 65.145 \), i.e. 9.8 cm/year and the maximum of burn-up \( BU_{\text{max}} = 10.82 \) at%. It is worth to note that the burn-up is not far from its maximum, if \( L \) is larger than a certain value, e.g. \( L > 15 \). As an example, the distribution of nuclide atom number densities is shown in Fig. 6 for the solitary wave solution.

Finally two remarks are addressed here. Being distinguished from the CANDLE concept, the subcriticality of fresh fuel is not required by this fundamental burn-up solution due to the fuel inlet boundary condition. This means that the fresh fuel can be supercritical and the function of \( k_c(\psi) \) can be monotonically decreasing or in particular no burnable poison is needed by this solution.

The burn-up of the pebble-bed reactor considered here is around 10 at%, which is several times less than the theoretical value for a fast CANDLE reactor. There are two reasons for the lower burn-up. First, we have chosen a fairly small enrichment for the fresh fuel. Higher enrichment, maybe together with burnable poison, would lead to higher fuel burn-up. Second, the typical cross-sections given in Table 1 are only valid for a thermal Maxwellian neutron spectrum, with a conversion ratio of only about 0.4 for the fresh fuel in our current example. If using more representative cross-sections, e.g. averaged over a spectrum of an existing HTGR, the conversion ratio for the fresh fuel might be close to or even slightly exceed unity, so that a higher fuel burn-up could be achieved.

4. Conclusion

By studying the one-dimensional diffusion-burn-up coupled neutron model, a family of analytic solutions has been found for the pebble-bed type reactor. This solution, called fundamental burn-up mode, has less restrictions on the fuel burn-up characteristics than the previous solitary burn-up wave solution and, therefore, extends the CANDLE burn-up concept to the more general cases of a core with finite dimensions and therefore non-zero boundary conditions. The solitary burn-up wave solution is only a particular case with a minimum fuel drift speed and a maximum burn-up.

References
