Lesson 13: Reactor Kinetics-2

- Step Change in Reactivity (Constant Reactivities)
  - Reactivity Equation (Inhour Equation)
- Roots of Reactivity Equation
- Stable Reactor Period (for positive reactivity)
  - Delayed, Prompt Criticality
- “Prompt Jump” Approximation
- Negative Reactivities
- Control Rod Calibration
Step Change in Reactivity (Constant Reactivity)

- Important specific application of point kinetics equations
  - Illustrative, analytical solution possible

- Constant $\rho$ (±) introduced at $t = 0$
  - E.g. quick movement of a control rod

- Point kinetics eqns.:

\[
\begin{align*}
0: & \quad \frac{dP(t)}{dt} = \frac{\rho - \beta}{\Lambda} \cdot P(t) + \sum_{i=1}^{6} \left[ \lambda_i \cdot C_i(t) \right] \\
1: & \quad \frac{dC_i(t)}{dt} + \lambda_i \cdot C_i(t) = \frac{\beta_i}{\Lambda} \cdot P(t)
\end{align*}
\]

- For solution, one may apply method of Laplace transforms
  - Differential equations replaced by algebraic eqns.
Laplace Transforms

- Useful method for study of time-dependent behaviour of a system
  - Differential eqn. (in time) converted to algebraic eqn. (in frequency)
- Transform: \( \mathcal{L} \left[ F(t) \right] = \int_0^\infty F(t) \cdot e^{-st} \, dt = \tilde{F}(s) \)
- After certain manipulations, one “reconverts” to obtain solution of original eqn. (in time)
- Often needed information got directly from algebraic eqn. itself (e.g. via graphical soln.)

- N.B.: not necessary to affect transformations analytically:
  - One can use “tables” of Laplace transforms

<table>
<thead>
<tr>
<th>( F(t) )</th>
<th>( \tilde{F}(s) )</th>
<th>( F(t) )</th>
<th>( \tilde{F}(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F'(t) )</td>
<td>( s\tilde{F}(s) - F(0) )</td>
<td>( 1 )</td>
<td>( 1/s )</td>
</tr>
<tr>
<td>( F''(t) )</td>
<td>( s^2\tilde{F}(s) - sF(0) - F'(0) )</td>
<td>( t )</td>
<td>( 1/s^2 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( t^{n-1/(n-1)!} )</td>
<td>( 1/s^n )</td>
</tr>
<tr>
<td>( \int_0^t F(t) , dt )</td>
<td>( \frac{1}{s^2} \tilde{F}(s) )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( -tf(t) )</td>
<td>( \tilde{F}'(s) )</td>
<td>( e^{-at} )</td>
<td>( 1/(s+a) )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( t e^{-at} )</td>
<td>( 1/(s+a)^2 )</td>
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</table>
Solution of Point Kinetics Eqns.

From \( \Theta \), \( \Theta \)

\[
\Rightarrow \quad \lambda \cdot \tilde{P}(s) - P(0) = \frac{\rho - \beta}{\Lambda} \cdot \tilde{P}(s) + \sum_{i=1}^{6} \left[ \lambda_i \cdot \tilde{C}_i(s) \right]
\]

\[
\lambda \cdot \tilde{C}_i(s) - C_i(0) + \lambda_i \cdot \tilde{C}_i(s) = \frac{\beta_i}{\Lambda} \cdot \tilde{P}(s)
\]

where \( \tilde{P}(s) \), \( \tilde{C}_i(s) \) are the Laplace transforms of \( P(t) \), \( C_i(t) \)

Eliminating \( \tilde{C}_i(s) \),

\[
\tilde{P}(s) = \Lambda \cdot \frac{P(0) + \sum_{i=1}^{6} \left[ \frac{\lambda_i \cdot C_i(0)}{\lambda_i + s} \right]}{\beta - \sum_{i=1}^{6} \left[ \frac{\beta_i \cdot \lambda_i}{\lambda_i + s} \right] + \Lambda s - \epsilon}
\]

If the reactor was critical at \( t = 0 \),

\[
\lambda_i \cdot C_i(0) = \frac{\beta_i \cdot P(0)}{\Lambda} \Rightarrow C_i(0) = \frac{\beta_i \cdot P(0)}{\lambda_i \cdot \Lambda}
\]
Solution of Point Kinetics Eqns. (contd.)

Thus, \[ \frac{\tilde{P}(\omega)}{P(0)} = \frac{\Lambda + \sum_{i=1}^{6} \left( \frac{B_i}{\lambda_i + \omega} \right)}{\beta - \sum_{i=1}^{6} \left( \frac{\beta_i \cdot \lambda_i}{\lambda_i + \omega} \right) + \Lambda \omega - \rho} \]

where \( \omega_j \) are the roots of the denominator

\[ \beta - \sum_{i=1}^{6} \left( \frac{\beta_i \cdot \lambda_i}{\lambda_i + \omega} \right) + \Lambda \omega - \rho = 0 \]

i.e. of

\[ \rho = \Lambda \omega + \sum_{i=1}^{6} \left( \frac{\beta_i \cdot \lambda_i}{\lambda_i + \omega} \right) \]

\[ \Rightarrow \rho = \Lambda \omega + \sum_{i=1}^{6} \left( \frac{\beta_i \cdot \omega}{\lambda_i + \omega} \right) \]

Reactivity Equation (Inhour Equation)

The sought solution:

\[ \frac{P(t)}{P(0)} = \sum_{j=1}^{7} \left[ B_j \cdot e^{\omega_j t} \right] \]

(from inversion of the Laplace-transform equation (1) above)
The Roots $\omega_j$

- Characteristic equation: 
  \[
  \rho = \Lambda \omega + \sum_{i=1}^{6} \left[ \frac{\beta_i \omega}{\lambda_i + \omega} \right]
  \]
  gives the relationship between $\omega$, $\rho$ (and the kinetics parameters $\Lambda$, $\beta_i$, $\lambda_i$)

- One may solve the equation (Reactivity Eqn.) graphically (Ligou considers: $\rho - \omega = ...$)

- R.H.S.... function of $\omega$
- L.H.S.... $\rho$ ($\pm$), constant

- For +ive $\rho$ (supercritical reactor), one value of $\omega$ is +ive, the others -ive

- For -ive $\rho$ (subcritical reactor), all 7 roots $\omega$ are -ive
Stable Period ... positive $\rho$

- For $\rho > 0$, after a certain time:
  \[ P(t) \approx B_1 \cdot e^{\omega_1 t} \]
  - All the other $e^{\omega_i t}$ terms disappear (negative $\omega_i$'s)

- Stable period: $T = \frac{1}{\omega_1}$
  - Time (in stable region) for $P$ (or $\Phi$) to increase by factor of $e$

- Solution $\omega_1$ obtained graphically
  - For a given set of kinetics parameters $(\Lambda, \beta_i, \lambda_i)$, each $\rho \leftrightarrow$ specific $\omega_1$
  - E.g. with

\[
\begin{align*}
\Lambda &= 10^{-3} \\
\beta_i, \lambda_i &\Rightarrow \text{U}^{235} \\
\rho &= 3 \cdot 10^{-3} \\
2 \omega_1 &= 0, 127 \text{ s}^{-1} \Rightarrow T = 7,94
\end{align*}
\]
Stable Period (contd.)

Without delayed n’s, one had:

\[ P(t) = P(0) \cdot \exp \left( \frac{k_{eff} - 1}{\frac{\lambda}{\rho}} \cdot t \right) = P(0) \cdot e^{\frac{\rho}{\lambda} \cdot t} \]

\[ T = \frac{\lambda}{\rho} = \frac{10^{-3}}{3 \cdot 10^{-3}} = 0.334 \]

\[ \Rightarrow \text{a factor of } \sim 24 \text{ on period (for this example)} \]

\[ \Rightarrow \text{Delayed n’s render the reactor controllable} \]

For “U^{235} systems” (\( \beta_i , \lambda_i \)):

For \( \rho \ll \beta_i \), no dependence on \( \Lambda \)

In absence of delayed n’s (\( \beta = 0 \)),

\[ \nu_0 = \frac{\rho}{\Lambda} \quad (T = \frac{\Lambda}{\rho}) \]

- For small \( \rho \)'s, differences with/without delayed n’s, very large

- For \( \rho \sim \beta \), differences decrease strongly (role of \( \Lambda \) becomes much more important)
Delayed, Prompt Criticality

- Normally, reactor is “delayed critical”:

\[ \text{Reactivity} = (1 - \beta) k_{\text{eff}} + \sum_i \left( \lambda_i c_i \right) \]

- If \( \rho = \beta \), reactor is critical only with prompt n’s (T becomes very short)

- For \( \rho >> \beta \), delayed neutrons no longer important
  - \( \rho \) vs \( \omega \) curves become asymptotic to \( \rho = \Lambda \omega + \beta \)

\[ \rho = \Lambda \omega + \sum_i \left[ \frac{\lambda_i \omega}{\lambda_i + \omega} \right] \approx \Lambda \omega + \beta \quad \text{for} \quad \lambda_i \ll \omega \]

- Extremely important that all reactivity insertions are significantly less than \( + \beta \)
  - Withdrawal of a control rod
  - Effects of temperature, voidage (e.g. due to boiling of a liquid moderator)

- \( \rho = \beta \) important enough limit to provide a special unit for reactivity, the \textit{dollar}
  - \( \rho = 0.65\% \) \textit{(U}^{235} \text{ system)} \Rightarrow 1 \text{ dollar} \ (100 \text{ cents})
  - \( \textit{For} \ \textit{U}^{233}, \textit{Pu}^{239}, \ 1\$ \sim 0.2 \text{ to } 0.3\% \)
Delayed, Prompt Criticality (contd.)

- For systems with Pu\textsuperscript{239}, U\textsuperscript{233}, reactivity insertions have to be smaller
  - It is \( \frac{\rho}{\beta} \) which matters, i.e. the value of \( \rho \) in \( \$ \) or \( \$ \)...

- Fast reactors (with Pu\textsuperscript{239} as primary fuel material), more sensitive because of their lower value of \( \beta \), not because of their very small \( \Lambda \)
  - However, \( \rho_{\text{abs}} \) values are also generally lower in fast reactors

Comments

- Normally, \( \rho \) change not sudden, e.g. could be ~ linear ramp in reactivity: \( \rho(t) = K.t \)
  - Point kinetics equations need to be solved numerically

- Often, changes in \( \rho \) not uniform... require consideration of spatial effects
  - Much more complicated eqns. than for Point Reactor model (which remains useful...)

- Change in power changes thermal balance, affects temperatures, hence \( \sigma \)'s and \( \rho \)
  - Feedback effects: have to be negative (act as “brakes”)... safety studies involve coupling of thermal-hydraulics, neutron kinetics
“Prompt Jump” Approximation

- One has seen... $\Phi \uparrow$ rapidly at beginning, before stable period is established
  - What is the rapid increase in flux value during the transitory phase?

- An approximation one can make sometimes:
  $\Rightarrow C_i$’s remain constant immediately after the $\rho$ change ("prompt jump" approxn.)
  i.e.
  \[
  \frac{dC_i}{dt} + \lambda_i C_i(0) = \frac{\lambda_i}{\Lambda} P(0)
  \]

Thus, from
\[
\frac{dP}{dt} = \frac{\rho - \beta}{\Lambda} P(t) + \text{Sum}_i \left[ \lambda_i C_i(t) \right]
\]

one has:
\[
\frac{dP}{dt} = \frac{\rho - \beta}{\Lambda} P(t) + \frac{\beta}{\Lambda} P(0)
\]

Solution:
\[
P(t) = \frac{\rho}{\rho - \beta} P(0) e^{\frac{\rho - \beta}{\Lambda} t} - \frac{\beta}{\rho - \beta} P(0)
\]

For $\rho \ll \beta$, exponential term decreases rapidly
\[
\Rightarrow P(t) \to \frac{\beta}{\beta - \rho} P(0)
\]

- "Prompt jump":
  \[
  \frac{P}{P(0)} \approx \frac{\beta}{\beta - \rho}
  \]
“Prompt Jump” Approxn. (contd.)

- Result valid for $\rho \ll \beta$, i.e. for small +ive $\rho$’s, as well as for all -ive $\rho$’s.
- Notions of stable period and “prompt jump” (or “prompt drop”, for -ive $\rho$’s) provide all information necessary for describing reactor response to sudden change in $\rho$ ($\ll \beta$).
- Considering $\Phi \sim P$:

\[
\frac{\Phi(t)}{\Phi(0)} = \frac{\beta}{\beta - \rho} \quad (t \rightarrow)
\]

\[
\Phi(t) = \Phi_0 \cdot e^{t/\tau}
\]

[Diagram showing the relationship between $\Phi(t)$ and $\Phi(0)$ with a prompt jump and stable region indicated]
For a “prompt drop”

For the stable period, one needs to consider the roots of the Reactivity Equation

- All the $\omega$ values are negative with $|\omega_1| > |\omega_2| > |\omega_3| > \ldots > |\omega_n|$
- The stable region is where only term $-\omega_1$ remains and the flux is:
  \[ \phi(t) = \phi_1 \cdot e^{-\lambda t} \quad \text{with} \quad T = \frac{1}{|\omega_1|} \]

For negative $\rho$ with $|\rho| \gg \beta$, $\omega_1 \to \lambda_1$ (decay const. of 1st gp.)

Thus, a reactor cannot be shut down more quickly than with a -ive period or $\sim 80s$

The “neutronic” power is determined by the prompt jump and $T$ (80s)

After a certain time, thermal power $\sim$ fission-product “decay power” (residual heat)
Calibration of a Control Rod

- Reactivity “worth” of a control rod… function of its insertion in the core
  
  For insertion $\Delta z$, effect $\sim C \cdot \Delta z \cdot [\phi(z)]^2$

  i.e.
  $$\frac{(\Delta k_{\text{eff}})_z}{(\Delta k_{\text{eff}})_{\text{tot}}} = \frac{\int_0^z [\phi(z)]^2 \, dz}{\int_0^H [\phi(z)]^2 \, dz}$$

  With $\phi = \phi_0 \cdot \sin\left(\frac{\pi z}{H}\right)$,

  $$(\Delta k_{\text{eff}})_z \Rightarrow \frac{z}{H \cdot \phi_0^2} \cdot \int_0^z [\phi(z)]^2 \, dz$$

  (“S-curve”)

- For calibrating a control rod, one may make a step $\Delta z$ (small +ive $\rho$) and measure stable period...

- Another method: “rod drop”, large -ive $\rho$ yielding:

  $$\frac{\phi_z}{\phi(0)} \approx \frac{\beta}{\beta - \epsilon} \Rightarrow \epsilon \approx \left[1 - \frac{\phi(0)}{\phi_z}\right] \cdot \beta$$
Summary, Lesson 13

- Reactivity Equation (constant reactivities)
- Roots of Reactivity Equation
  - Stable Reactor Period
- Delayed, prompt criticality
  - Dollar
- “Prompt jump” (small +ive, all -ive ρ’s)
- Negative reactivities
- Control rod calibration