Lesson 9: Multiplying Media (Reactors)

- Multiplication Factors
- Reactor Equation for a Bare, Homogeneous Reactor
- Geometrical, Material Buckling
- Spherical, Slab Reactors
- Cylindrical Reactor
- Absolute Value of Flux
- Maximum-to-Average Flux Ratio
- Comments on Criticality Condition
Media Containing Fuel

- Neutron propagation considered in passive media till now
  - Characterised by $\Sigma_a$, $\Sigma_t$ (or $\Sigma_s$), ...
  - External neutron source(s)

- For a medium containing fissile material, one has a source per unit volume:

$$Q_f(\vec{r}) = \bar{\nu} \cdot \Sigma_f(\vec{r}) \cdot \phi(\vec{r})$$

  - arising from fissions

  - Here, $Q_f$ is not a part of the data provided, but rather is dependent on the flux, i.e. on the sought solution itself

- The neutronics description yields one or several homogeneous equations
  - Stationary solution exists only if a certain criticality condition is satisfied by the geometrical and material characteristics of the system
Media Containing Fuel (contd.)

- For the critical state: productions = absorptions + leakage
  - Condition ~ independent of the flux
    (all three “reaction rates” vary in about the same proportions)

- We will first consider a reactor which is homogenous and bare
  - Isolated zone, without a “reflector” or a “blanket” (i.e. just the “core”)

- Treatment will first be one-group
  - Monoenergetic neutrons (e.g. thermal with appropriately averaged cross-sections)

- Later will be considered
  - Multizone reactors (e.g. with a reflector, i.e. an outer zone of pure moderator)
  - Multigroup theory (basis used, in most practical cases, for numerical calculations)
  - Certain aspects of the “heterogeneity” of the reactor lattice
    (heterogeneous unit-cell: fuel/moderator/...
Multiplication Factors

- For the infinite medium, in general

\[
k_\infty = \frac{\text{production}}{\text{absorption}} = \frac{\bar{\Sigma} \cdot S_f}{A}
\]

with

\[
\begin{align*}
\bar{S} &= \int \varphi \varphi' \, dv \\
A &= \Sigma_a \int \varphi \varphi' \, dv
\end{align*}
\]

\[
= \bar{\Sigma} \cdot \frac{\Sigma_f}{\Sigma_a}
\]

with

\[
\begin{align*}
\Sigma_f &= (\Sigma_f)_e \\
\Sigma_a &= (\Sigma_a)_e + (\Sigma_a)_p
\end{align*}
\]

\[
\eta_c = \left( \bar{\Sigma} \cdot \frac{\Sigma_f}{\Sigma_a} \right)_e \quad \text{and} \quad f = \frac{(\Sigma_a)_e}{(\Sigma_a)_e + (\Sigma_a)_p}
\]

\[
\eta_c > 1 \quad \text{and} \quad f < 1
\]
Multiplication Factors (contd.)

- $\eta_c \Rightarrow$ fuel multiplication factor
- $f \Rightarrow$ utilisation factor (for the homogeneous mixture)

\[
f = \frac{(\Sigma_a)_c}{(\Sigma_a)_c + (\Sigma_a)_f}
\]

\[
= \frac{N_c \sigma_{ac}}{N_c \sigma_{ac} + N_m \sigma_{am} + N_g \sigma_{ag}}
\]

moderator

structure (cladding, etc.)

- For a heterogeneous unit-cell, one needs to consider that the flux is depressed within the fuel region…
System of Finite Size

- For the infinite medium: \( k_\infty = \eta_c \cdot f \)
- This corresponds to \( k_\infty = \eta_c \cdot f \cdot \epsilon \cdot p \), with \( \epsilon = p = 1 \)
  - All the neutrons have the same energy
    - There are neither resonance absorptions, nor fast fissions
- For the finite system, one has the effective multiplication factor

\[
\eta^{\text{production}} = \frac{\eta^{\text{absorption + leakage}}}{A + F} = \frac{\eta^{\text{absorption}}}{A + F} = \frac{\eta^{\text{production}}}{A} \cdot \left( \frac{A}{A + F} \right)
\]

\( k_{\text{eff}} = k_\infty \cdot P_{NF} \)

where \( P_{NF} = \frac{A}{A + F} \) \( \Rightarrow \) Non-leakage probability
System of Finite Size (contd.)

- Three cases to be considered:
  - \( k_{\text{eff}} = 1 \) \( \Rightarrow \) critical reactor (self-sustaining chain reaction, constant flux)
  - \( k_{\text{eff}} > 1 \) \( \Rightarrow \) supercritical reactor (divergent reaction, increasing flux)
  - \( k_{\text{eff}} < 1 \) \( \Rightarrow \) subcritical reactor (convergent reaction, decreasing flux)

- \( k_{\text{eff}} \) \( \Rightarrow \) most important single parameter for the functioning of a reactor

- Important to note: \( k_{\text{eff}} < k_\infty < \eta_c < \bar{\nu} \)
Equation for the Critical Reactor

- One uses the 1-group diffusion equation for the stationary case (critical reactor):
  \[ D \nabla^2 \phi - \Sigma_a \phi + Q = 0 \]

- Thereby:
  \[ Q = Q_f = \vec{\Sigma}_f (S_f / \chi) = \vec{\Sigma}_f \cdot R_f = \vec{\Sigma}_f \cdot \Sigma_f \phi \]

- Thus,
  \[ D \nabla^2 \phi + (\vec{\Sigma}_f - \Sigma_a) \phi = 0 \quad \Rightarrow \quad \nabla^2 \phi + \left[ \frac{\vec{\Sigma}_f}{\Sigma_a} - \frac{1}{\Sigma_a} \right] \phi = 0 \]

- Eqn. homogeneous
  - Can be solved for a given geometry,
  - Condition to be satisfied can be identified

- Simplest systems... homogeneous spherical reactor, slab reactor

- One-group Reactor Equation
- Material Buckling
  (depends only material properties)
Spherical Reactor

- One has

\[
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d\Phi}{d\rho} \right) + B^2_m \Phi = 0
\]

with \( B^2_m = \frac{k_{o-1}}{L^2} \)

(positive sign, cf. passive medium with point source)

Using \( \Phi(\rho) = \frac{x(\rho)}{\rho} \),

\[
\frac{d^2x}{d\rho^2} + B^2_m x = 0
\]

\[
x(\rho) = A \sin B_m \rho + C \cos B_m \rho
\]

, i.e.

\[
\Phi(\rho) = A \cdot \frac{\sin B_m \rho}{\rho} + C \cdot \frac{\cos B_m \rho}{\rho}
\]

- For the finite system:

1. \( \Phi \neq \infty \) at \( \rho = 0 \) \( \Rightarrow \) \( C = 0 \)

2. \( \Phi = 0 \) at \( \rho = R+d = R + 0.71 \lambda_t \), i.e. \( \sin \{B_m (R+d)\} = 0 \)
Spherical Reactor (contd.)

- From the condition \( \sin \{B_m(R+d)\} = 0 \):
  \[ B_m = B_i = \frac{i\pi}{R + d} \text{ for } i = 1, 2, \ldots \]

- For the critical reactor, only \( i = 1 \) is valid
  - Smallest eigenvalue
    \( \Rightarrow \) **Fundamental Mode**

- Other solutions: higher harmonics
  - Exist only in a subcritical system
  - E.g. near the external source

- Critical condition for the spherical reactor is thus:
  \[ B_m^2 = B^2 = \left( \frac{\pi}{R + d} \right)^2 \]
  Geometrical Buckling
  (depends only on system dimensions)
Spherical Reactor (contd.2)

- The flux distribution is:
  \[ \phi(\varepsilon) = \frac{A}{\varepsilon} \sin \left( \frac{\pi \varepsilon}{R + d} \right) \]

- For a given medium (specific values of \( B_m^2, d \)), R is determined by the criticality condition
  - Critical radius:
    \[ R_c = \left( \frac{\pi}{B_m} \right) - d \]
  - Critical mass:
    \[ M_c = \frac{4}{3} \pi R_c^3 \cdot \rho \]

- Conversely, if the size (R) is fixed, the material properties need to be identified which yield the appropriate \( B_m^2 \) (material buckling)…
  - E.g. adjust the enrichment, i.e. \( k_\infty \)
Comments

- For a bare homogeneous reactor, the criticality condition demands an eigenvalue search for the reactor equation:
  \[ \nabla^2 \phi + B_m^2 \phi = 0 \]
  - Eigenvalues need to go to zero at the extrapolated outer surface

- The square of the smallest eigenvalue: \(B^2\) (geometrical buckling)

- Criticality condition: \(B_m^2 = B^2\)

- Flux distribution: proportional to the eigenfunction corresponding to \(B^2\)
  - Absolute level of the flux not yet known
  - A constant \(\lambda\) (in the example considered) remains undetermined, depends on the power (determined, in turn, by technological constraints…)

*Neutronics allows us to determine the criticality condition and the spatial distribution of the flux, but does not fix the latter’s absolute value…*
Slab Reactor

- System infinite in the \( X, Y \) directions
- Height \( H \)
- Flux, function only of \( z \)

Reactor Equation:

\[
\frac{d^2 \phi}{dz^2} + B^2 \phi = 0
\]

General solution:

\[
\phi(z) = A \cos Bz + C \sin Bz
\]

Flux, symmetric relative to plane \( z = 0 \) \( (d\phi/dz = 0 \text{ at } z = 0) \)

\Rightarrow \quad C = 0
Thus, \( \phi(z) = A \cos Bz \)

Condition \( \Phi = 0 \) at \( \frac{H}{2} + d \) yields as eigenvalues:

\[
B_i = (2i+1) \cdot \left[ \frac{\pi}{2 \left( \frac{H}{2} + d \right)} \right]
\]

with \( i = 0, 1, 2, \ldots \)

Square of the smallest eigenvalue (\( i = 0 \))

\[
B^2 = \left( \frac{\pi}{H+2d} \right)^2
\]

(geometrical buckling)

Flux distribution:

\[
\phi(z) = A \cos \left( \frac{\pi z}{H+2d} \right)
\]
Cylindrical Reactor

For a cylindrical reactor of height $H$ and radius $a$, the Critical Reactor Equation is:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

With the assumption $\phi(r,z) = R(r) \cdot Z(z)$,

$$\frac{1}{R} \left[ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -B^2$$

- $\downarrow$ function of $r$
- $\downarrow$ function of $z$
- $-\alpha^2$
- $-\beta^2$

constant
Thus,

\[ \frac{d^2 Z}{dz^2} + \alpha^2 Z = 0, \quad \cdots (1) \]

& \quad B^2 = \alpha^2 + \beta^2 \quad \cdots \left\{ \begin{array}{c} \alpha^2 \to \text{axial buckling} \\ \beta^2 \to \text{radial buckling} \end{array} \right. 

From (1), \quad Z(k) = A' \cos \alpha z, \quad \text{and with} \quad \phi = 0 \quad \text{at} \quad L = \frac{k}{2} + d, \quad \alpha = \frac{\pi}{H+2d} \quad \text{(smallest eigenvalue)}

From (2), \quad R(r) = A'' \left[ J_0(\beta r) \right] \quad (\gamma \text{ not applicable... } \phi \neq -\infty) 

\text{and with} \quad \phi = 0 \quad \text{at} \quad r = a+d, \quad \beta = \frac{2.405}{a+d} \quad \text{(smallest eigenvalue)}
Cylindrical Reactor (contd.2)

Thus,
\[ \phi(r, \theta) = A \cdot J_0 \left( \frac{2.405r}{a+d} \right) \cdot \cos \left( \frac{\pi z}{H+2d} \right) \]

and
\[ B^2 = \alpha^2 + \beta^2 = \left( \frac{2.405}{a+d} \right)^2 + \left( \frac{\pi}{H+2d} \right)^2 \]

Comments:

The criticality condition implies:
\[ B^2 = B_m^2 = \frac{k_{\infty} - 1}{\lambda^2} \] (material buckling)

(i) If \( H \) is so small that \( \left( \frac{\pi}{H+2d} \right)^2 > B_m^2 \), even \( a \to \infty \) does not suffice...
\[ \Rightarrow \text{System remains subcritical} \quad (B^2 > B_m^2, \ldots) \]

(ii) Similarly, if \( a \) is too small, \( \left( \frac{2.405}{a+d} \right)^2 > B_m^2 \)
\[ \Rightarrow \text{Even with } H \to \infty, \quad B^2 > B_m^2, \quad k_{\text{eff}} < 1 \ldots \]
Cylindrical Reactor (contd.3)

(iii) There are minimal values for $H$, $a$

\[
\alpha_{\text{max}}^2 = \left( \frac{\pi}{H_{\text{min}} + 2d} \right)^2 = B_m^2
\]
\[\Rightarrow H_{\text{min}} = \frac{\pi}{B_m} - 2d\]

\[
\beta_{\text{max}}^2 = \left( \frac{2.405}{a_{\text{min}} + d} \right)^2 = B_m^2
\]
\[\Rightarrow a_{\text{min}} = \frac{2.405}{B_m} - d\]

(iv) Many combinations of $H$, $a$ for the critical state
- Shaded area corresponds to supercritical states
  (e.g. power reactor at “start-of-cycle”)
Absolute Flux value

- For a spherical reactor, if one neglects $d$

$$\phi(e) \approx \frac{A \sin \left(\frac{\pi e}{R}\right)}{e}, \quad \text{i.e.} \quad R \approx R_e = R + d$$

"extrapolated radius"

- Constant $A$ is determined by the reactor power

$$P = \int_0^R E_f \cdot \Sigma_f \cdot \phi(e) \cdot 4\pi e^2 \, de = 4\pi A \cdot E_f \cdot \Sigma_f \int_0^R e^2 \sin \left(\frac{\pi e}{R}\right) \, de$$

$$= 4\pi A \cdot E_f \cdot \Sigma_f \cdot \frac{R^2}{\pi}$$

$$\Rightarrow A = \frac{P}{4R^2 \cdot E_f \cdot \Sigma_f}$$

$$\Rightarrow \phi(e) = \frac{P}{4R^2 \cdot E_f \cdot \Sigma_f} \cdot \left[ \frac{\sin \left(\frac{\pi e}{R}\right)}{e} \right]$$
Absolute Flux value (contd.)

- For a slab reactor

$$\phi(z) \approx A \cos \left( \frac{\pi z}{H} \right)$$

with

$$H \sim H_e = H + 2d$$

Per cm\(^2\) of the slab,

$$P'' = \int_{-H/2}^{H/2} E_f \cdot \Sigma_f \cdot \phi(z) \cdot (1. dz)$$

watts/cm\(^2\)

$$= E_f \cdot \Sigma_f \cdot \int_{-H/2}^{H/2} A \cos \left( \frac{\pi z}{H} \right) dz$$

$$= A \cdot E_f \cdot \Sigma_f \cdot \frac{e^H}{\pi}$$

$$\Rightarrow A = \frac{\pi P''}{2H \cdot E_f \Sigma_f}$$

$$\Rightarrow \phi(z) = \frac{\pi P''}{2H \cdot E_f \Sigma_f} \cdot \cos \left( \frac{\pi z}{H} \right)$$

- For a cylindrical reactor, one can show:

$$\Rightarrow \phi(z) = \frac{2.405 \pi P}{4 \cdot E_f \Sigma_f \cdot J_0 \left( \frac{2.405r}{a} \right)} \cdot J_0 \left( \frac{2.405z}{a} \right) \cdot \cos \left( \frac{\pi z}{H} \right)$$
(\frac{\Phi_{\text{max}}}{\overline{\Phi}}) \text{ Ratio}

- The flux distribution determines the power distribution
  - Reactor homogeneous... constant $\Sigma_f$

- The maximum-to-average ratio, same for flux, power

- For bare, homogeneous reactors, flux always maximum at centre, varies strongly (going to zero at extrapolated surface)
  - In practice, one has a reflector and/ or several different zones in the reactor, which render the flux distribution flatter...
(\( \Phi_{\text{max}} / \Phi \)) \text{ Ratio (Examples)}

\( \Phi_{\text{max}} = \Phi \) \quad \text{at centre of system (bare, homogeneous reactor)}

\[
\Phi = \frac{1}{V} \int \Phi \, dV = \frac{1}{V} \cdot \frac{P}{E_f \Sigma_f}
\]

**Spherical Reactor**

\[
\Phi_{\text{max}} \rightarrow \lim_{\rho \to 0} \left[ \frac{A \sin \frac{\pi \rho}{R}}{\rho} \right] = \frac{A \pi}{R} = \frac{\pi P}{4 R^2 E_f \Sigma_f}
\]

\[
\Phi = \frac{P}{V \cdot E_f \Sigma_f} \quad \frac{P}{3 \pi R^2 \cdot E_f \Sigma_f} = \frac{3 P}{4 R^2 E_f \Sigma_f}
\]

\[
\Phi_{\text{max}} \rightarrow \frac{\pi^2}{3} \cong 3.29
\]

**Slab Reactor**

\[
\lim_{\rho \to 0} \left[ A \cos \frac{\pi \rho}{H} \right] = A = \frac{\pi^2 P''}{2 H \cdot E_f \Sigma_f}
\]

\[
\frac{P''}{(1-H) \cdot E_f \Sigma_f} = \frac{P''}{H \cdot E_f \Sigma_f}
\]

\[
\frac{\pi}{2} \cong 1.57
\]
Comments on Criticality Condition

\[ B_n^2 = \frac{k_{\infty} - 1}{L^2} = B^2 \quad \text{material buckling} \]
\[ = \text{geometrical buckling} \quad \left| B^2 \left(\frac{R_e}{R_c}\right)^2 \right| \text{ sphere etc.} \]

(a) Rewriting above equation:
\[ k_{\infty} = 1 + L^2 B^2 \quad \ldots \ (1) \]

Previously, one had:
\[ k_{\text{eff}} = k_{\infty} \cdot P_{\text{NF}} \cdot \frac{\text{Abs.}}{\text{Abs. + Leakage}} \quad \ldots \ (2) \]

From (1) (criticality condition):
\[ k_{\text{eff}} = \frac{k_{\infty}}{1 + L^2 B^2} = 1 \quad \ldots \ (3) \]

Comparing (2), (3):
\[ P_{\text{NF}} = \frac{1}{1 + L^2 B^2} \]
Comments on Criticality Condition (contd.)

(b) In general, for given values of \( k_\infty \), \( B^2 \):

One may nevertheless use the formalism of a critical system (stationary flux)…

- We consider a fictitious medium in which the number of n’s produced per fission

\[
B_{n*}^2(k_{\text{eff}}) = \frac{k_\infty / k_{\text{eff}} - 1}{L^2}
\]

For this medium,

\[
B_{n*}^2 = B^2
\]

yields

\[
k_{\text{eff}}^* = \frac{k_\infty / k_{\text{eff}}}{1 + L^2 B^2} = 1
\]

(criticality condition)

For the actual system, one needs to “search” for the modification which leads to

- Change in \( B^2 \) or in \( k_\infty \).

For a bare homogeneous reactor, the search is direct…

For a multizone system (heterogeneous layout), one needs to adopt an iterative approach

- The dimensions (or the material characteristics) are varied until

\[
k_{\text{eff}} = k_{\text{eff}}^* = 1.
\]
(c) We have

$$k_0 = \frac{\Sigma_e}{\Sigma_a}, \quad L^2 = \frac{D}{\Sigma_a}$$

$$\Rightarrow k_{\text{eff}} = \frac{k_0}{1 + L^2 B^2} = \frac{\Sigma_e}{\Sigma_a + \Sigma_a B^2} - \frac{\text{Prod.}}{\text{Abs.} + \text{Leakage}}$$

Comparing this with:

$$k_0 = \frac{\Sigma_e}{\Sigma_a} - \frac{\text{Prod.}}{\text{Abs.}}$$

$$\Rightarrow \text{Leakage term } DB^2 \text{ is like a supplementary } \Sigma_a \ldots$$

One may consider a reactor of finite dimensions (with a geometrical buckling of $B^2$), as though it were an infinite medium with a “poisoning” of $\Sigma_a = DB^2 \ldots$

(However, reactor equation still needs to be solved – for obtaining $B^2 \ldots$)
Summary, Lesson 9

- Multiplying media, multiplication factors
- 1-group, diffusion equation (Reactor Eq.)
- Material and geometrical buckling
- Bare homogeneous reactors (sphere, slab, cylinder, etc.)
- Absolute flux and reactor power
- Maximum- to-average flux ratio
- Comments on criticality condition
  - Non-leakage probability, criticality “search”, leakage as “absorptions” (DB²)